THERE ARE NO NONCOMMUTATIVE SOFT MAPS

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Abstract. It is shown that for a map $f\colon X \to Y$ of compact spaces the unital $*$-homomorphism $C(f)\colon C(Y) \to C(X)$ is projective in the category $\mathrm{Mor}(\mathcal{C}^1)$ precisely when X is a dendrite and f is either homeomorphism or a constant.

1. Introduction

By Gelfand's duality any topological property of a categorical nature in the category $\mathcal{COMP}$ (= compact spaces and their continuous maps) has its counterpart in the category $\mathcal{AC}^1$ (= commutative unital C^*-algebras and their unital $*$-homomorphisms) which, in turn, serves as a prototype for the corresponding concept in the larger category $\mathcal{C}^1$ (= unital C^*-algebras and their unital $*$-homomorphisms).

For example, X is an injective object in $\mathcal{COMP}$ (i.e. X is a compact absolute retract) precisely when the C^*-algebra $C(X)$ is a projective object in $\mathcal{AC}^1$. However, requirement that $C(X)$ is actually projective object in the full category $\mathcal{C}^1$ imposes severe restrictions back on X: as shown in [3] this happens if and only if X is a dendrit (i.e. at most one dimensional metrizable AR-compactum). In other words, the class of dendrits coincides with the class of noncommutative absolute retracts.

Expanding further to the category $\mathrm{Mor}(\mathcal{COMP})$ we note that injective objects in it are also well understood and play important role in geometric topology. These are soft maps between AR-compacta. Recall (see, for instance, [1, Definition 2.1.33]) that a map $f\colon X \to Y$ of compact spaces is soft if for any compact space B, any closed subset $A \subset B$, and any two maps $g\colon A \to X$ and $h\colon B \to Y$ such that $f \circ g = h|A$, there exists a map $k\colon B \to X$ such that $g = k|A$ and $f \circ k = h$. Here is the diagram illustrating the situation:

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, hook, "\mathrm{incl}"'] & X \arrow[d, "f"] \\
B \arrow[r, "h"] \arrow[ur, dotted, "k"] & Y
\end{tikzcd}
$$

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As noted, by reversing arrows and allowing all (not necessarily commutative) unital C^*-algebras, we arrive to the following concept of doubly projective homomorphism. This concept was first introduced in [4, Definition 3.1] and studied also in [2]. It must be noted that in [2], as well as below, we do not assume (while [4] does) that the domain of doubly projective homomorphism is projective.

Definition 1.1. A unital $*$-homomorphism $i\colon X \to Y$ of unital C^*-algebras is doubly projective if for any unital $*$-homomorphisms $f\colon X \to A$, $g\colon Y \to B$ and any surjective unital $*$-homomorphism $p\colon A \to B$ with $g \circ i = p \circ f$, there exists a unital $*$-homomorphism $h\colon Y \to A$ such that $f = h \circ i$ and $g = p \circ h$. In other words, any commutative diagram (of unbroken arrows)

$$\begin{tikzcd} B & Y \arrow[l, "g"'] \arrow[dl, dotted, "h"'] \\ A \arrow[u, "p"] & X \arrow[l, "f"] \arrow[u, "i"'] \end{tikzcd}$$

with surjective p can be completed by the dotted diagonal arrow with commuting triangles.

Lemma 1.1. *Retract of a doubly projective homomorphism is doubly projective. More precisely, suppose that $i_1\colon X_1 \to Y_1$ is doubly projective, and for a unital $*$-homomorphism $i_2\colon X_2 \to Y_2$ there exist unital homomorphisms $s_1\colon X_2 \to X_1$, $r_1\colon X_1 \to X_2$ with $r_1 \circ s_1 = \mathrm{id}_{X_2}$ and $s_2\colon Y_2 \to Y_1$, $r_2\colon Y_1 \to Y_2$ with $r_2 \circ s_2 = \mathrm{id}_{Y_2}$. If $i_1 \circ s_1 = s_2 \circ i_2$, then i_2 is also doubly projective.*

Proof. Consider the following diagram of unbroken arrows

$$\begin{tikzcd} B & Y_2 \arrow[l, "g"'] \arrow[dl, dotted, "h"'] \\ A \arrow[u, "p"] & X_2 \arrow[l, "f"] \arrow[u, "i_2"'] \end{tikzcd}$$

with p is surjective as in Definition 1.1. We need to construct a dotted $*$-homomorphism h making both triangular diagrams commutative.

Since i_1 is doubly projective there exists a unital $*$-homomorphism $q\colon Y_1 \to A$ such that $g \circ r_2 = p \circ q$ and $f \circ r_1 = q \circ i_1$. Here is the full diagram

$$
\begin{tikzcd}
B & Y_2 \arrow[l, "g"'] \arrow[r, bend right, "s_2"'] \arrow[dl, dotted, "h"'] & Y_1 \arrow[l, "r_2"'] \arrow[dll, dotted, "q"] \\
A \arrow[u, "p"] & X_2 \arrow[l, "f"'] \arrow[u, "i_2"'] \arrow[r, bend right, "s_1"'] & X_1 \arrow[l, "r_1"'] \arrow[u, "i_1"']
\end{tikzcd}
$$

Let $h = q \circ s_2$. It only remains to note that

$$f = f \circ r_1 \circ s_1 = q \circ i_1 \circ s_1 = q \circ s_2 \circ i_2 = h \circ i_2$$

and

$$g = g \circ r_2 \circ s_2 = p \circ q \circ s_2 = p \circ h.$$

□

Theorem 1.2. *Let $f\colon X \to Y$ be a surjective map of a compact space X onto a non-trivial Peano continuum Y. If $C(f)\colon C(Y) \to C(X)$ is doubly projective, then f is a homeomorphism.*

Proof. Assume the contrary and let $y_0 \in Y$ be point such that $|f^{-1}(y_0)| > 1$. Since $C(f)$ is doubly projective in the category $\mathcal{C}^1$ it is doubly projective in the smaller category $\mathcal{A}C^1$. By Gelfand's duality the latter means precisely that f is a soft map. Choose points $x_0, x_1 \in f^{-1}(y_0)$ with $x_0 \neq x_1$. Softness of f guarantees that there exist two sections $i_0, i_1\colon Y \to X$ of f such that $i_k(y_0) = x_k$ for each $k = 0, 1$. Note that the set $V = \{y \in Y\colon i_0(y) \neq i_1(y)\}$ is a non-empty (since $y_0 \in V$) open subset of Y and in view of our assumption contains a homeomorphic copy of the segment $[0,1] \subset V$ (i.e. geodesic segment in V between two points - denoted by 0 and 1). Let $Z = f^{-1}([0,1])$ and fix a retraction $r\colon Y \to [0,1]$. Since $f|Z\colon Z \to [0,1]$ is soft there exists a retraction $s\colon X \to Z$ such that $f \circ s = r \circ f$. Then, by Lemma 1.1, $C(f|Z)\colon C([0,1]) \to C(Z)$ is doubly projective. Since $C([0,1])$ is projective in $\mathcal{C}^1$ we conclude by [2, Lemma 5.3] that $C(Z)$ is projective in $\mathcal{C}^1$. Consequently, by [3, Theorem 4.3], Z is a dendrite, in particular, $\dim Z = 1$. Consider the fiber $f^{-1}(0) \subset Z$. Since f is soft, $f^{-1}(0)$ is a non-trivial absolute retract and consequently contains a segment $[i_0(0), i_1(0)]$ connecting the points $i_0(0)$ and $i_1(0)$. Similarly, fiber $f^{-1}(1)$ contains a segment $[i_0(1), i_1(1)]$ connecting the points $i_0(1)$ and $i_1(1)$. Clearly the union S of these four segments $[i_0(0), i_1(0)]$, $i_1([0,1])$, $[i_0(0), i_1(0)]$ and $i_0([0,1])$ is homeomorphic to the circle S^1. Since $\dim Z = 1$, there exists retraction $p\colon Z \to S$. But this is impossible because Z is an absolute retract. □

Corollary 1.3. *Let $f\colon X \to Y$ be a map of compact spaces. Then the following conditions are equivalent:*

(i) $C(f)\colon C(Y) \to C(X)$ *is a projective object of the category* $\mathrm{Mor}(\mathcal{C}^1)$*;*
(ii) X *is a dendrit and* f *is either a homeomorphism or a constant map.*

Proof. (i) $\Longrightarrow$ (ii). General nonsense easily implies that both $C(X)$ and $C(Y)$ are projective in $\mathcal{C}^1$. Thus, by [3], X and Y are dendrits. Also [2, Proposition 5.11] guarantees that $C(f)$ is doubly projective. By 1.2, f is either constant or a homeomorphism.

(ii) $\Longrightarrow$ (i) is trivial. □

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